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NOTE

The Permanent Rank of a Matrix

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1. INTRODUCTION

Recall that the permanent of an $n \times n$ matrix is defined like the determinant, but with all $n!$ terms having positive signs (see [8]). Define the *perrank* of a matrix A to be the size of the largest square submatrix of A with nonzero permanent. Throughout this paper F denotes a field, p denotes char(F) and Z_p denotes the finite prime field of p elements. All matrices have entries in F. We assume $p \neq 2$ throughout. Of course when $p = 2$ permanent and determinant coincide.

LEMMA 1.1. *Suppose A is an $n \times m$ matrix with perrank$(A) = m < n$. Let $(A\ \ x)$ be the $n \times (m+1)$ matrix formed by A following with the column vector x. Then dim$\{x\colon$ perrank$(A\ \ x) = m\} \leqslant m$.*

Proof. It is clear that $L := \{x\colon \text{perrank}(A\ \ x) = m\}$ is a subspace. We may assume that the submatrix of A formed by its first m rows has nonzero permanent. Then since for every $x \in L$ the perrank of $(A\ \ x)$ is not full, the projection of L to the first m coordinates is one to one. Hence dim$(L) \leqslant m$. ∎

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In this lemma if we replace perrank by rank, then $\dim(L)=m$; so by induction on m we have

COROLLARY 1.2. *Let $m \leqslant n$ and A be an $n \times m$ random matrix with each entry chosen independently and uniformly from a finite field F. Then $Pr(perrank(A)=m) \geqslant Pr(rank(A)=m)$.*

In particular, if $m=n$ we have $\Pr(\mathrm{per}(A)=0) \leqslant \Pr(\det(A)=0)$. As $n \to \infty$, $\lim(\Pr(\det(A)=0))$ is well-known (see e.g. [4]). But as far as we know, $\lim(\Pr(\mathrm{per}(A)=0))$ is not known.

THEOREM 1.3. *For any matrix A, $perrank(A) \geqslant rank(A)/2$.*

Proof. We may assume A is an $n \times n$ nonsingular matrix and let $m = \mathrm{perrank}(A)$. There exists an $n \times m$ submatrix B with full perrank. Lemma 1.1 gives $n-m \leqslant \dim\{x: \mathrm{perrank}(B\ \ x)=m\} \leqslant m$. ∎

Theorem 1.3 is tight as shown by $\left(\begin{smallmatrix} I & I \\ I & -I \end{smallmatrix}\right)$. This is essentially the only tight example we know at this point.

On the other hand, for $p>0$ perrank is also bounded above in terms of rank: it is easy to see that $\mathrm{perrank}(A) \leqslant (p-1)\,\mathrm{rank}(A)$. The key is, if a square matrix has a column (or row) repeating p times, then its permanent is 0. This simple and known observation also gives

LEMMA 1.4. *Suppose A is a matrix with a column repeating $p-1$ times, $p>0$. Then adding a multiple of this column to any other column doesn't change $perrank(A)$.*

This lemma suggests that we can use elementary operations to study the permanent. (This idea has also been discovered independently by Kogan and Makowsky; see Section 3.5 of [7].) The following result (true for $p=3$ only) is an application of this idea.

THEOREM 1.5. *If $p=3$, then $perrank(A) = perrank(A^{-1})$ for any nonsingular A.*

Proof. Let $T_1, T_2, T_3, \ldots$ be a sequence of elementary column operations that change $A_{n \times n}$ to I. Consider the sequence of operations T_1, T'_1, T_2, T'_2, T_3, T'_3, ... on $\left(\begin{smallmatrix} I & A \\ I & A \end{smallmatrix}\right)$, where T_i is applied to the left n columns, and T'_i is the corresponding operation applied to the right n columns. We do operations in pairs so that every column in each intermediate matrix is repeated.

By Lemma 1.4, $\mathrm{perrank}\left(\begin{smallmatrix} I & I \\ A & A \end{smallmatrix}\right) = \mathrm{perrank}\left(\begin{smallmatrix} A^{-1} & A^{-1} \\ I & I \end{smallmatrix}\right)$. Set $C = \left(\begin{smallmatrix} I & I \\ A & A \end{smallmatrix}\right)$. It suffices to show $\mathrm{perrank}(C) = n + \mathrm{perrank}(A)$.

On the one hand, if B is an $r \times r$ submatrix of A with nonzero permanent, then the submatrix of C generated (in the obvious sense) by the left upper I and the right lower B has nonzero permanent $2^r \operatorname{per}(B)$. So $\operatorname{perrank}(C) \geqslant n + \operatorname{perrank}(A)$.

On the other hand let D be a square submatrix of C with nonzero permanent. Suppose D has $r+s$ rows from $(I \;\; I)$ with r of them having two ones, s of them having a single one; and D has t rows from $(A \;\; A)$. The general form of D is:

$$D = \begin{pmatrix} I_r & I_r & 0 & 0 \\ 0 & 0 & I_s & 0 \\ U_{t \times r} & U_{t \times r} & V_{t \times s} & W_{t \times (t-r)} \end{pmatrix}$$

Here U, V are submatrices of A and W is a submatrix of $(A \;\; A)$. Write $W = (W_1 \;\; W_2)$ where W_1, W_2 are submatrices of A and W_1 has at least $(t-r)/2$ columns. It is clear that $(U \;\; V \;\; W_1)$ is a submatrix of A, so in particular $n \geqslant r + s + (t-r)/2$, or equivalently $(t+r)/2 \geqslant r + s + t - n$. Since $\operatorname{per}(U \;\; W) = 2^{-r} \operatorname{per}(D) \neq 0$, we have $\operatorname{perrank}(U \;\; W_1) \geqslant (t+r)/2 \geqslant r + s + t - n$. So $\operatorname{perrank}(C) \leqslant n + \operatorname{perrank}(A)$. ∎

2. THE ALON–JAEGER–TARSI CONJECTURE

The Alon–Jaeger–Tarsi Conjecture, first proposed in 1981, is

Conjecture 2.1. *For any field F with $|F| \geqslant 4$ and any nonsingular matrix A over F, there is a vector x such that both x and Ax have only nonzero entries.*

This was proved by Alon–Tarsi [3] for every non-prime finite field, but the case $F = Z_p$ ($p \geqslant 5$) remains open. (The case $|F| = \infty$ is trivial.) See [9] for a matroid approach. Motivated by Conjecture 2.1, Jeff Kahn (personal communication) proposed:

Conjecture 2.1. *For any $n \times n$ nonsingular matrix A over any field, $\operatorname{perrank}(A \;\; A) = n$.*

Conjecture 2.2 implies Conjecture 2.1 via the polynomial argument of [3]. It also immediately implies Theorem 1.3. Actually Theorem 1.3 was motivated by Conjecture 2.2. Conjecture 2.2 is true when $p = 3$ (see [3]) and we also verified it for $n \leqslant 4$.

Theorem 2.3. *For any $n \times n$ nonsingular matrices $A_1, A_2, ..., A_l$,*

$$\operatorname{perrank}(A_1 \;\; A_2 \;\; \cdots \;\; A_l) \geqslant (1 - 2^{-l})n$$

Proof. Theorem 1.3 gives $\text{perrank}(A_1) \geqslant n/2$ and Lemma 1.1 implies $\text{perrank}(A_1\ A_2\ \cdots\ A_{k+1}) \geqslant (\text{perrank}(A_1\ A_2\ \cdots\ A_k)+n)/2$. The theorem follows by induction on l. ∎

This theorem implies a small result on Conjecture 2.1. Choose $l=p-2$ and $A_1=A_2=\cdots=A_l=A$. Then the argument of [3] gives

PROPOSITION 2.4. *For any $n\times n$ nonsingular matrix A over Z_p, there is a vector x without zero entries such that Ax has at most $n/2^{p-2}$ zero entries. In particular, Conjecture 2.1 is true if $n<2^{p-2}$.*

We don't know examples of equality in Theorem 2.3 if $l\geqslant 2$ and we believe the theorem is not tight. In fact, we don't know any example with $\text{perrank}(A_1\ A_2)<n$. Motivated by the additive basis problem, the authors of [2] conjectured that $\text{perrank}(A\ B)=n$ for any nonsingular $n\times n$ matrices A, B over Z_3. (*Remarks*: this statement can be shown by Lemma 1.4 to be equivalent to Conjecture 4.3 (in the case $p=3$) of [2]. The additive basis problem was motivated by the nowhere-zero flow problem, see Section 3.3 of [5] for more details.)

Noga Alon (personal communication) has shown that for A chosen uniformly from the nonsingular $n\times n$ matrices over F, Conjecture 2.1 is true almost surely as $n\to\infty$. We will show that Conjecture 2.2 is true almost surely in the same sense.

LEMMA 2.5. *For an $m\times n$ matrix A with $\text{perrank}(A\ A)=m<2n$,*

$$dim\left(L:=\left\{x\colon perrank\begin{pmatrix} x & x\\ A & A\end{pmatrix}=m\right\}\right)\leqslant\frac{m}{2}$$

(*where of course x runs over row vectors of length n*).

Proof. We regard the columns of A as distinct even though A may have identical columns, so that we can label the column set of A by $[n]$. Then the column set of $(A\ A)$ is naturally the multiset $W:=[n]\cup[n]$. For any $U\subseteq W$, define $|U|$ to be the cardinality of U, $\|U\|$ to be the number of repeated elements in U and $A(U)$ to be the $m\times|U|$ submatrix of $(A\ A)$ consisting of the columns in U. For example, $|W|=2n$, $\|W\|=n$, $A([n])=A$.

Choose $V\in\{U\colon U\subseteq W, |U|=m, \text{per}(A(U))\neq 0\}$ such that $\|V\|$ is maximum. We may assume that $V=[r]\cup[s]$ (multiset union), where $r=\|V\|$, $0\leqslant r\leqslant s$, $r+s=m$.

It suffices to show that the projection of L to the first r coordinates is one to one, since $r\leqslant m/2$. So assume $x\in L$ and $x_1=x_2=\cdots=x_r=0$. It is in fact enough to show that $x_{r+1}=x_{r+2}=\cdots=x_s=0$, since $x=0$ is then immediate (This is as in the proof of Lemma 1.1).

Assume $r<s$ (otherwise we are done). Let $V'=V\cup\{r+1\}$. Since

$$\|V'\backslash\{r+2\}\|=\|V'\backslash\{r+3\}\|=\cdots=\|V'\backslash\{s\}\|=r+1>\|V\|,$$

we have $\operatorname{per}(A(V'\backslash\{r+2\}))=\cdots=\operatorname{per}(A(V'\backslash\{s\}))=0$ by our choice of V.
Since $x\in L$, the $(m+1)\times(m+1)$ submatrix of $\left(\begin{smallmatrix} x & x \\ A & A\end{smallmatrix}\right)$ consisting of the columns in V' has zero permanent. This means exactly that $x_{r+1}=0$ (using $p\neq 2$). Similarly $x_{r+2}=\cdots=x_s=0$. ∎

The next corollary gives Conjecture 2.2 almost surely for a uniformly chosen random nonsingular matrix, since $\lim(\Pr(\det(A)\neq 0))>0$ (again see [4]).

Corollary 2.6. *Suppose A is an $n\times n$ random matrix with each entry chosen independently and uniformly from a finite field F, then $Pr(perrank(A\ A)=n)\to 1$ as $n\to\infty$.*

Proof. Let A_i be an $i\times n$ random matrix and $q=|F|$. It is clear $\Pr(\operatorname{perrank}(A_1\ A_1)=1)=(1-q^{-n})$. By Lemma 2.5 we have $\Pr(\operatorname{perrank}(A_2\ A_2)=2)\geqslant(1-q^{-n})^2$. By induction (and again Lemma 2.5) we have for $i\leqslant n$,

$$\Pr(\operatorname{perrank}(A_{2i}\ A_{2i})=2i)\geqslant(1-q^{-n})^2(1-q^{-n+1})^2\cdots(1-q^{-n+i-1})^2$$

Applying this with $i=[n/2]+1$ yields the corollary. ∎

We mention that Lemma 2.5 can be generalized. Suppose $(A\ A\ \cdots\ A)$, with A repeated k times, has full perrank $m<kn$, and $p>k$ or $p=0$, then with L defined in analogy with Lemma 2.5, $\dim(L)\leqslant m/k$. We can use this and Lemma 1.4 to give an alternative proof of (essentially) the bound in Lemma 3.6 of [2].

Finally we have a strange result for fields of characteristic 0.

Lemma 2.7. *Suppose the ith row of $A_{m\times n}$ has at least $2i-1$ nonzero elements and $p=0$, then $perrank(A)=m$.*

Proof. We proceed by induction, the lemma being trivial when $m=1$. We label the column set of A by $[n]$ and let B be the submatrix of A formed by its first $m-1$ rows. Any $(m-1)$-set U of $[n]$ corresponds to an $(m-1)\times(m-1)$ submatrix of B and we use $\operatorname{per}(U)$ to denote its permanent. Since dividing a column by a nonzero number or permuting columns doesn't affect the problem, we may assume that the last row of A is $(1, 1, ..., 1, 0, 0, ..., 0)$ with k ones.

Now suppose perrank$(A) < m$. For any fixed $(m-1)$-set $U \subseteq [n]$, let $R = U \cap [k]$, $r = |R|$ and $V = U \backslash R$. For any $(r+1)$-set $T \subseteq [k]$, the $m \times m$ submatrix of A corresponding to $T \cup V$ has zero permanent, that is

$$\sum_{S \subseteq T,\ |S| = r} \operatorname{per}(S \cup V) = 0$$

We have an unknown per$(S \cup V)$ for each r-set $S \subseteq [k]$ and we have an equation for each $(r+1)$-set $T \subseteq [k]$. The coefficient matrix of this linear system is exactly the inclusion matrix $M(k, r, r+1)$. Kantor [6] showed that inclusion matrix always has full rank over field of characteristic 0. Since $k \geqslant 2m-1 > 2r$, the number of equations is at least the number of unknowns, so all unknowns are zero. In particular, per$(U) = 0$, that is, perrank$(B) < m-1$, a contradiction. ▮

This lemma gives the first part of the following theorem, and the second part then follows via the argument of [3] and Theorem 1.2 of [1].

THEOREM 2.8. *Suppose the ith row of $A_{n \times n}$ has at least i nonzero elements and $p = 0$, then* (1) *perrank* $(A \quad A) = n$;

(2) *for any $S_1, S_2, ..., S_n \subseteq F$ with $|S_i| = 3$, there are $x_i \in S_i$ such that Ax has no zero entries.*

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